THE 3-SPACE $PG(3, 2)$ AND ITS GROUP.*

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Introduction. An n-space in which the points are determined by homogeneous coordinates will contain only a finite number of points if the coordinates are restricted to be integers reduced to a modulus. In this paper the modulus 2 is used, i. e., the only numbers possible are 0 and 1.

The configuration of the "real" points, i. e., those whose coordinates are expressed in terms of 0 or 1, is studied in detail.

The complete projective group of collineations and dualities of the 3-space is shown to be of order $|\underline{8}$ and to have as a sub-group the linear homogeneous group L. H. G. $\overset{15}{\frac{|\underline{8}}{2}}$, this sub-group consisting of the projective collineations of the 3-space. The line coordinates for the lines of the 3-space are introduced and the linear complexes and congruences obtained. These line coordinates can also be considered as the coordinates of points in a 5-space. To every transformation of the 3-space there corresponds a transformation of the 5-space. In the 5-space, there are determined 8 sets of 7 points each, "heptads," by means of which is established the isomorphism of the linear homogeneous group L. H. G. $\overset{15}{\frac{|\underline{8}}{2}}$ with the alternating group on 8 letters.

An 8 letter notation is derived from the "heptads" for the points, lines, and planes of the 3-space.

The geometry gives a complete solution of Kirkman's School Girl Problem and is related to several functions which are of importance in the Galois theory of equations. The configuration of the 3-space is the same as that studied by Moore in this connection.†

The Configuration of the "Real" Points. A point is determined in a 3-space by 4 homogeneous coordinates (a, b, c, d) and when the numbers

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*Oswald Veblen and W. H. Bussey, Finite Projective Geometries. *Transactions of the American Mathematical Society*, vol. 7 (1906), pp. 241–259.

† E. H. Moore, Concerning the General Equations of the Seventh and the Eighth Degrees *Mathematische Annalen*, vol. 51 (1899), pp. 417–444,

a, b, c, d, are integers and reduced modulo 2 we obtain 15 points excluding the combination (0, 0, 0, 0) ; these are called the "real" points of the 3-space.

If $P_{a_1} = (a_1, b_1, c_1, d_1)$ and $P_{a_2} = (a_2, b_2, c_2, d_2)$ are any two points, the points of the line joining them are given by

$$P_{\lambda a_1 + \eta a_2} = (\lambda a_1 + \eta a_2, \lambda b_1 + \eta b_2, \lambda c_1 + \eta c_2, \lambda d_1 + \eta d_2) ;$$

the only possible sets of values for (λ, η) are (1, 0), (0, 1), (1, 1), hence the number of points on a line is 3. The points of a line may thus be denoted by P_{a_1}, P_{a_2}, $P_{a_1+a_2}$. Consider a fourth point $P_{a'}$, which is not on the line P_{a_1}, P_{a_2}, $P_{a_1+a_2}$; evidently $P_{a_1+a'}$, $P_{a_2+a'}$, $P_{a_1+a_2+a'}$, are points of the lines joining $P_{a'}$ to the points of the line P_{a_1}, P_{a_2}, $P_{a_1+a_2}$.

These 7 points are all the points of a plane, since any 2 points P_x and P_y determine a collinear point P_{x+y}, which is contained among the 7. The configuration of the 7 points of a plane is that of a complete quadrangle in which the diagonal points are collinear. The number of lines in a plane is 7 as there are 7 points and 3 lines through each point.

In the 3-space the number of "real" planes is the same as the number of "real" points, since for each possible set of point coordinates there is a possible set of plane coordinates. Consider the 7 points of a plane which can be denoted by

$$P_{a_1},\ P_{a_2},\ P_{a_1+a_2},\ P_{a'},\ P_{a_1+a'},\ P_{a_2+a'},\ P_{a_1+a_2+a'},$$

and a point Pa'' not upon the plane ; this will determine 7 other points

$$P_{a_1+a''}, P_{a_2+a''}, P_{a_1+a_2+a''}, P_{a'+a''}, P_{a_1+a'+a''}, P_{a_2+a'+a''}, P_{a_1+a_2+a'+a''}.$$

These 15 points constitute all the "real" points of the 3-space, for any 2 points P_x and P_y determine a collinear point P_{x+y}, which is included among the 15 and any 3 non-collinear points P_x, P_y, P_z determine the 7 points of a plane, which can be denoted by

$$P_x,\ P_y,\ P_{x+y},\ P_z,\ P_{x+z},\ P_{y+z},\ P_{x+y+z}.\text{*}$$

Any 4 non-coplanar points determine 15 points. This process of "doubling" from a point without an n-space always leads to an $(n + 1)$-space or from a $PG(n, 2)$ to a $PG(n + 1, 2)$.

Through each point there are 7 lines, for take any point and a plane not

* E. H. Moore, loc. cit.

containing it, the 7 lines joining this point to the points of a plane contain all 15 points. The number of lines is 35, since there are 15 points and 7 lines through each point and 3 points on a line: $7.15/3 = 35$.

Of the 15 planes 3 pass through each line. Consider a plane and a point outside the plane, the point with the 7 lines of the plane determines 7 distinct planes, one of which contains any 2 points of the 3-space, hence there are 7 planes through each point.

The whole configuration can be exhibited in the table *

	S_0	S_1	S_2
S_0	15	7	7
S_1	3	35	3
S_2	7	7	15

S_0 is a point, S_1 a line, S_2 a plane and in general S_π is an n-space. A number n in the ith row and the ith column gives the number of i-spaces in the given space, while a number n in the ith column and the kth row gives the number of i-spaces which are united with a k-space.

By the use of the transformation T of period 15

$$T \quad \begin{aligned} x_1' &= x_1 + \quad\; + \quad\; x_4 \\ x_2' &= x_1 + x_2 + \quad\; x_4 \\ x_3' &= x_1 + x_2 + x_3 \\ x_4' &= x_1 + x_2 + x_3 + x_4 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= x_1' \quad\; + x_3' + x_4' \\ x_2 &= x_1' + x_2' \\ x_3 &= \quad\; x_2' \quad\; + x_4' \\ x_4 &= \quad\; + x_3' + x_4' \end{aligned}$$

all the points can be obtained from a single one, say (1, 1, 1, 1), and all the planes from a single one, say $x_1 + x_2 + x_3 + x_4 = 0$.

1	(1,1,1,1)	I	$x_1 + x_2 + x_3 + x_4 = 0$
2	(0,1,1,0)	II	$x_4 = 0$
3	(0,1,0,0)	III	$x_3 + x_4 = 0$
. . . . .		. . . .	

* E. H. Moore, Tactical Memoranda I-III. *American Journal of Mathematics*, vol. 18 (1896), pp. 264-303.

Hence denoting the points by the numbers 1 to 15, the point i is obtained from the point 1 (1,1,1,1) by the transformation T^i. All the planes are obtained from one expressed through its 7 points as 1 (1,2,5,7,12,13,14) by the transformation T (1,2,3,4, · · ·, 14,15) and its powers.

And all the lines can be obtained from 3, each expressed in terms of its 3 points, (1, 2, 5), (1, 3, 9), (1, 6, 11), by means of the same transformation T. The first two lines have each 15 conjugates while the third has 5 under the transformation T and its powers.

Introduction of Line Coordinates. Two points P_a and P_a' of the 3-space determine a line: P_α (a_1, a_2, a_3, a_4) and $P_\alpha'(a_1', a_2', a_3', a_4')$.

There are 12 determinants p_{ik} of the form

$$p_{ik} = \begin{vmatrix} a_i & a_\kappa \\ a_i' & a_\kappa' \end{vmatrix}$$

which are related two by two, $p_{i\kappa} = p_{\kappa i}$ (since $p = -p$ modulo 2).

The 6 quantities $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ are the Plücker line coordinates and completely determine a line. They are related by the condition

$$p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23} = 0.*$$

The line coordinates of the 35 lines can be obtained from those of the 3 lines (1, 2, 5), (1, 3, 9), (1, 6, 11) by means of a transformation P on the p's which is induced by the transformation T:

$$\begin{aligned}
p_{12}' &= p_{12} & &+ p_{24} \\
p_{13}' &= p_{12} + p_{13} + p_{14} & &+ p_{24} + p_{34} \\
p_{14}' &= p_{12} + p_{13} & &+ p_{24} + p_{34} \\
p_{23}' &= \quad + p_{13} + p_{14} + p_{23} & &+ p_{24} + p_{34} \\
p_{24}' &= \quad + p_{13} \quad + p_{23} & &+ p_{34} \\
p_{34}' &= \quad\quad p_{14} + & &+ p_{24} + p_{34}
\end{aligned}$$

The 3 lines and their line coordinates are respectively (1, 2, 5), (1, 3, 9), (1, 6, 11), and (1,1,0,0,1,1), (1,0,0,1,1,0), (0,1,0,1,0,1).

It is evident that the line coordinates may be looked upon as points in a

*C. M. Jessop, *Treatise on the Line Complex*, page 17.

5-space, a $PG(5, 2)$. In such a space there are $2^6 - 1 = 63$ points excluding $(0,0,0,0,0,0)$. Of these 63 points, the 35 representing lines of the 3-space lie upon a surface, S,

$$p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23} = 0.$$

The lines whose coordinates satisfy a linear equation constitute a linear complex, which can be either degenerate or non-degenerate.

A degenerate complex consists of all the lines which meet a given line, called the axis, together with this axis. Given two lines p and p' with the line coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ and $(p'_{12}, p'_{13}, p'_{14}, p'_{23}, p'_{24}, p'_{34})$, the condition that they may intersect is

$$p_{12}p'_{34} + p_{13}p'_{24} + p_{14}p'_{23} + p_{23}p'_{14} + p_{24}p'_{13} + p_{34}p'_{12} = 0.$$

Hence this condition, if the p's are considered constant, is the equation of a degenerate complex, consisting of all lines meeting p' and of p' itself. The equation is satisfied for p' since the equation then reduces to the condition S.

Any linear relation among the p's, as

$$ap_{12} + bp_{13} + cp_{14} + dp_{23} + ep_{24} + fp_{34} = 0,$$

can be looked upon as the polar of the point (f,e,d,c,b,a) with respect to the surface S and if this point in the 5-space represents a line in the 3-space, then the complex determined will be degenerate; if it does not represent a line, the complex is non-degenerate. Hence the polar of a point on the surface S, i. e., the tangent 4-space at the point to the surface S, has in common with the surface S the points representing the lines of a degenerate complex, while the polar of a point off the surface S determines in a similar way a non-degenerate complex. The number of degenerate complexes is equal to the number of points in the 5-space which represent lines in the 3-space, that is 35. They may all be obtained from the following three by the transformation P and its powers:

$$p_{12} + p_{13} + p_{24} + p_{34} = 0, \quad p_{12} + p_{23} + p_{34} = 0, \quad p_{13} + p_{23} + p_{34} = 0.$$

A non-degenerate complex is a system of lines such that through every point of the 3-space there is a flat pencil of lines and all the lines in every plane pass through a point. The proof is the same as in the ordinary case.*

* Jessop, loc. cit., page 25.

The lines whose coordinates satisfy two linear equations and hence are common to two complexes, constitute a congruence. Consider the two complexes C and C':

$$ap_{12} + bp_{13} + cp_{14} + dp_{23} + ep_{24} + fp_{34} = 0,$$
$$a'p_{12} + b'p_{13} + c'p_{14} + d'p_{23} + e'p_{24} + f'p_{34} = 0;$$

these determine a third complex, $C'' = C + C'$,

$$(a + a')\,p_{12} + (b + b')\,p_{13} + (c + c')\,p_{14} + (d + d')\,p_{23} + (e + e')\,p_{24} + (f + f')\,p_{34} = 0.$$

C, C' and C'' are degenerate if the points (a,b,c,d,e,f), (a',b',c',d',e',f') and $(a + a', b + b', c + c', d + d', e + e', f + f')$ respectively are upon the surface S. The condition that this last point is on the surface reduces to

$$af' + a'f + be' + b'e + cd' + c'd = 0.$$

This is the condition that the axes of C and C' intersect. Hence if the three complexes of a family are all degenerate, the congruence determined consists of 11 lines meeting the two intersecting axes of the complexes C and C'. If two of the complexes of the family are degenerate, while the third is non-degenerate, the congruence consists of 9 lines meeting the two non-intersecting axes of the degenerate complexes. If only one of the complexes of the family is degenerate, then the non-degenerate complexes contain the axis of the degenerate complex, the congruence consists of 7 lines, the axis and 6 lines meeting it, one line through each point of the 3-space. This is a degenerate congruence with an axis. When the three complexes of a family are all non-degenerate, there is a non-degenerate congruence determined, which consists of 5 lines such that there is one and but one through each point of the 3-space.

The Group of the 3-space. Every projective collineation* in the 3-space $PG(3, 2)$ is represented by a transformation T on the point coordinates

$$T: \quad x'_r = \sum_{i=1}^{i=4} a_{ri}x_i \qquad r = (1, 2, 3, 4).$$

* O. Veblen and W. H. Bussey, loc. cit., p. 253.

This transformation is completely determined if we know into what points the vertices of the tetrahedron of reference are transformed; (1,0,0,0) goes into $(a_{11}, a_{21}, a_{31}, a_{41})$, (0,1,0,0) goes into $(a_{12}, a_{22}, a_{32}, a_{42})$, etc.

In order that the transformation may have an inverse, the determinant of the transformation must be different from zero, and this is the condition that the four points $(a_{1i}, a_{2i}, a_{3i}, a_{4i})$, $(i = 1,2,3,4)$ into which the vertices of the tetrahedron are transformed shall not be coplanar. Hence the number of transformations of the group of projective collineations is equal to the number of ways four non-coplanar points can be chosen, i. e., $15 \cdot 14 \cdot 12 \cdot 8 = 20{,}160 = \lfloor 8/2$. The dualities or polarities of the 3-space are transformations of the form

$$x_r = \sum_{i=}^{i=4} a_{ri} u, \qquad r = (1, 2, 3, 4),$$

where the x's are point coordinates and the u's plane coordinates.

These transformations have the property of changing points into planes and vice versa, but change lines into lines. The number of these dualities is evidently the same as the number of projective collineations $\lfloor 8/2$. The order of the complete projective group, consisting of all projective collineations and dualities, is $\lfloor 8$.

Since a transformation of the complete projective group changes lines into lines, every transformation of this group determines a transformation on the line coordinates. Conversely every transformation on the line coordinates which changes lines into lines corresponds to a transformation of the complete projective group. Let the transformation on the p's be of the form

$$p'_r = \sum_{i=1}^{i=6} a_{ri} p_i, \qquad r = (1, 2, 3, 4, 5, 6),$$

The lines with the coordinates (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), (0,1,1,0,0,0), (1,0,1,0,0,0) and (1,1,0,0,0,0), all pass through the point (1,0,0,0) and must be transformed into lines, hence we must have

$$(A) \quad a_{1i}a_{6i} + a_{2i}a_{5i} + a_{3i}a^{4i} = 0, \quad (i = 1, 2, 3),$$

$$(B) \quad (a_{1i} + a_{1j})(a_{6i} + a_{6j}) + (a_{2i} + a_{2j})(a_{5i} + a_{5j}) + (a_{3i} + a_{3j})(a_{4i} + a_{4j}) = 0,$$

(B) reduces by means of (A) to (B') :

$$(B') \quad a_{1i}a_{6j} + a_{1j}a_{6i} + a_{2i}a_{5j} + a_{2j}a_{5i} + a_{3i}a_{4j} + a_{3j}a_{4i} = 0, \; (ij = 1, 2, 3).$$

(B') is the condition that the three lines (1,0,0,0,0,0), (0,1,0,0,0,0), (0,0,1,0,0,0), which are not in a plane, must be transformed into lines which meet two by two, hence they must pass through a point or lie in a plane. Hence all the lines through a point must be transformed into lines through a point, a projective collineation, or be transformed into the lines of a plane, a duality. The simplest transformation on the p's which is a duality is

$$p_1' = p_6, \quad p_2' = p_5, \quad p_3' = p_4, \quad p_4' = p_3, \quad p_5' = p_2, \quad p_6' = p_1.$$

The Configuration of the 28 Points not upon the Surface S in the 5-space. Interpreting the results of section 2 in terms of the 5-space we see that taking the polar of a point in the 5-space with respect to the surface S determines a non-degenerate complex for the 3-space. Taking the polars of three points of a line in the 5-space determines a non-degenerate congruence for the 3-space. Hence the number of non-degenerate congruences is the same as the number of lines in the 5-space, which are wholly off the surface S. Whatever is true for a single point of the 28 is true for all, as any point can be transformed into any other. The number of lines wholly off the surface S which can be drawn through the point (1,0,0,0,0,1) is 6 and the same number can be drawn through each of the 28 points, hence there are $28 \cdot 6/3 = 56$ lines wholly off the surface S. The number of non-degenerate congruences is thus 56.

Consider the 3 points of a line as $A = (1,0,0,0,0,1)$, $B = (1,0,1,1,0,0)$, $C = (0,0,1,1,0,1)$, which are wholly off the surface S. Through each of these points there are 5 other lines wholly off the surface S. These lines are denoted below in terms of their 3 points, the 3 points in a row are the 3 points of a line.

Lines through A	Lines through B	Lines through C
(1,0,0,0,0,1) (1,0,1,1,0,0) (0,0,1,1,0,1)	(1,0,0,0,0,1) (1,0,1,1,0,0) (0,0,1,1,0,1)	(1,0,0,0,0,1) (1,0,1,1,0,0) (0,0,1,1,0,1)
(1,0,0,0,0,1) (0,1,1,1,0,1) (1,1,1,1,0,0)	(0,1,1,1,0,1) (1,0,1,1,0,0) (1,1,0,0,0,1)	(1,1,1,1,0,0) (1,1,0,0,0,1) (0,0,1,1,0,1)
(1,0,0,0,0,1) (1,1,0,0,1,0) (0,1,0,0,1,1)	(0,1,0,0,1,1) (1,0,1,1,0,0) (1,1,1,1,1,1)	(1,1,1,1,1,1) (1,1,0,0,1,0) (0,0,1,1,0,1)
(1,0,0,0,0,1) (0,0,1,1,1,1) (1,0,1,1,1,0)	(0,1,1,1,1,1) (1,0,1,1,0,0) (1,0,0,0,1,1)	(1,0,0,0,1,1) (1,0,1,1,1,0) (0,0,1,1,0,1)
(1,0,0,0,0,1) (1,1,1,0,1,0) (0,1,1,0,1,1)	(0,1,0,1,1,0) (1,0,1,1,0,0) (1,1,1,0,1,0)	(0,1,0,1,1,0) (0,1,1,0,1,1) (0,0,1,1,0,1)
(1,0,0,0,0,1) (0,1,0,1,1,1) (1,1,0,1,1,0)	(1,1,0,1,1,0) (1,0,1,1,0,0) (0,1,1,0,1,0)	(0,1,1,0,1,0) (0,1,0,1,1,1) (0,0,1,1,0,1)

It will be observed that when two points as A and B of the line ABC are chosen, that each of the 5 other lines through A is met by a single line ot the 5 other lines through B. The 5 points determined by the intersection of these pairs of lines together with the 2 points A and B we designate a "heptad." It will be found that the $7\cdot6/2 = 21$ lines joining these 7 points by pairs are wholly off the surface S. If any 2 points of the heptad are chosen and these points are used in the same way as A and B for the determination of a heptad, the same heptad will be determined. Each of the pairs of points AB, BC, CA determines a heptad so that each point of the 28 points off the surface S belongs to 2 heptads. The number of heptads is thus $28\cdot2/7 = 8$. No 6 of the points of a heptad are in the same 4-space, no 5 in the same 3-space, no 4 in the same 2-space, no 3 in the same 1-space.

If we take the generators of the group of projective collineations of the 3-space, the $L.H.G \left|{}^{15}_{2} 8\right.$, the matrices below are the matrices of the 6 generators of the group.*

E_1	E_2	E_3	E_4	E_5	E_6
1110	0101	0110	1110	0101	0011
0111	0110	0011	0100	0011	0110
1111	0010	1011	0010	1111	0010
1011	1110	1111	0101	1011	1001

From these can be calculated the corresponding transformations in terms of the p's. Let the 8 heptads be given the numbers 1 to 8; each heptad is determined by two of its points.

(1,0,1,1,0,0) and (1,1,1,1,1,1) determine heptad 1

(0,0,1,1,0,0) and (1,1,1,0,1,0) determine heptad 2

(1,1,0,0,1,0) and (0,1,0,1,1,1) determine heptad 3

(0,0,1,1,1,1) and (0,1,1,1,0,0) determine heptad 4

(1,0,1,0,1,1) and (0,1,1,1,0,1) determine heptad 5

(1,0,0,0,0,1) and (1,0,1,1,0,0) determine heptad 6

(1,1,1,0,0,1) and (0,0,1,1,0,0) determine heptad 7

(0,0,1,1,1,0) and (1,1,0,0,1,0) determine heptad 8.

* E. H. Moore, loc. cit., page 435.

The transformations in terms of the p's applied to the 28 points off the surface S permute the heptads among themselves. To the 6 generators above correspond the permutations on the heptads,

$$E_1 = (4,7)(7,6),\ E_2 = (4,7)(6,8),\ E_3 = (4,7)(8,5),$$
$$E_4 = (4,7)(1,5),\ E_5 = (4,7)(1,2),\ E_6 = (4,7)(2,3);$$

and these are the generators of the alternating group on 8 letters.

Hence the $L.H.G \left| \overset{15}{8} \right._{2}$ and the alternating group on 8 letters are isomorphic.*

A transformation on the p's which is a duality is equivalent to a transformation of the symmetric group on 8 letters, for example, the duality of page 67 is equivalent to the transformation $(2,7)(3,6)(4,5)$ on the heptads. To the fact that by the addition of one duality to the group of projective collineations the complete projective group can be generated, corresponds the fact that by the addition of one uneven transformation to the alternating group the symmetric group is generated. The complete projective group of the 3-space, $PG(3,2)$, is isomorphic with the symmetric group on 8 letters.

By means of the heptads, it is possible to assign an 8 letter notation to the points of the 5-space which are off the surface S, and to study their configuration. We have pointed out the fact that each point off the surface S belongs to 2 heptads. The notation for a point can thus be taken as (a,b), $a \neq b$, and there are $8 \cdot 7/2 = 28$ combinations of this form corresponding to the 28 points off the surface S. The notation for the points of a line off the surface S is $(ab)(bc)(ca)$, a, b, c distinct; for the points of a line off the surface S by pairs determine three heptads. Thus the two letter notations for the points must by pairs have a letter in common. There are thus $8 \cdot 7 \cdot 6/1 \cdot 2 \cdot 3 = 56$ lines off the surface S. The maximum number of points which a plane can have off the surface S is six. Such a plane in terms of its points has the notation $(ab)(bc)(ca)(ad)(bd)(cd)$. A line off the surface S has the notation $(ab)(bc)(ca)$. For any point off the surface S has the notation (xy) and in order that the line determined by this point with (ab) may have its third point off the surface S, x or y must equal a or b since (ab) and (xy) must belong to a common heptad. Let $x = a$ then $y = d$, $d \neq a \neq b \neq c$. Then the two other

*First proved by Jordan, *Traité des substitutions*, No. 516. E. H. Moore has given a proof based on the same system of generators, loc. cit., page 432.

points of the plane which are off the surface S are (bd) and (cd). There are $8 \cdot 7 \cdot 6 \cdot 5 / 1 \cdot 2 \cdot 3 \cdot 4 = 70$ planes of this kind.

In a similar manner the 3-spaces and the 4-spaces with the maximum number of points off the surface S can be determined. The configuration of the n-spaces with the maximum number of points off the surface S is given below in a table whose interpretation is the same as that of the table page 62

	S_0	S_1	S_2	S_3	S_4
S_0	28	6	15	20	15
S_1	3	56	5	10	10
S_2	6	4	70	4	6
S_3	10	10	5	56	3
S_4	15	20	15	6	28

The Configuration of the Thirty-Five Points upon the Surface S in the 5-space. The surface S has as its equation

$$p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23} = 0$$

and is satisfied by the 35 points of the 5-space representing lines in the 3-space. Consider any two points upon the surface S, $(a_1, b_1, c_1, d_1, e_1, f_1)$ and $(a_2, b_2, c_2, d_2, e_2, f_2)$ the third point of the line, $(a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2, e_1 + e_2, f_1 + f_2)$ will be upon the surface S if

$$(a_1 + a_2)(f_1 + f_2) + (b_1 + b_2)(e_1 + e_2) + (c_1 + c_2)(d_1 + d_2) = 0$$

or if
$$a_1f_2 + a_2f_1 + b_1e_2 + b_2e_1 + c_1d_2 + c_2d_1 = 0.$$

This latter is the condition that the two lines in the 3-space represented by the first two points intersect. Any two points of a line can be taken as the first two, hence the 3 lines in the 3-space which are represented by the points of a line in the 5-space must meet by pairs in the 3-space and thus lie in a plane, or pass through a point.

Let $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$, $C = (c_1, c_2, c_3, c_4)$ be three points not on a line in the 3-space. The line coordinates of AB and AC are

$$(a_1b_2 + a_2b_1, \cdots, a_3b_4 + a_4b_3) \text{ and } (a_1c_2 + a_2c_1, \cdots, a_3c_4 + a_4c_3).$$

These line coordinates considered as coordinates of two points in the 5-space determine a collinear point,

$$(a [b_2 + c_2] + a_2[b_1 + c_1], \cdots, a_3[b_4 + c_4] + a_4[b_3 + c_3])$$

whose coordinates are the line coordinates of the line joining A to the third point of BC, $(b_1 + c_1, b_2 + c_2, b_3 + c_3, b_4 + c_4)$. Hence the three points of a line in the 5-space which is entirely on the surface S represent a flat pencil, three lines in the 3-space which lie in a plane and pass through a point.

The seven lines in a plane of the 3-space are represented by the 7 points of a plane wholly on the surface S in the 5-space, since in the plane in the 3-space every line meets every other line. In a similar manner the lines through a point in the 3-space are represented by the points of a plane which is wholly on the surface S in the 5-space.

The tangent 4-space at a point of the surface S has already been shown to have 19 points in common with the surface S, representing the 18 lines which meet a given line in the 3-space, and the line itself.

Since through a given line in the 3-space there are 3 planes and through each point in a plane 3 lines, it follows that the 19 points which the tangent 4-space has in common with the surface S are upon 9 lines through the point of tangency.

Any 4-space passing through the point of tangency has in common with the tangent 4-space a tangent 3-space. These tangent 3-spaces are of two kinds. The first kind has in common with the surface S 11 points, which are upon 5 lines through the point of tangency, representing a degenerate congruence determined by two degenerate complexes whose axes have a point in common. The second kind has in common with the surface S 7 points, which are upon 3 lines through the point of tangency; these represent the 7 lines of a degenerate congruence with an axis.

The tangent 2-spaces are determined by two 4-spaces through the point of tangency and by the tangent 4-space. They are of 5 kinds. One and two have 7 points in common with the surface S and are on 3 lines through the point of tangency. They represent in the 3-space the lines of a plane and the lines through a point respectively. Three has 5 points in common with the surface S, which are on 2 lines through the point of tangency and represent in the 3-space a line and a flat pencil through each of two points of the line determining two planes through the line. Four has three points of a line in common with the surface S and represents a flat pencil of three lines in the 3-space. Five is a tangent plane which has but a single point in common with the surface S and represents a single line in the 3-space.

In the configuration of the 28 points off the surface S there are 70 planes, which have 6 points off the surface S. Two of these planes pass through each of the 35 points of the surface S. We have seen that the notation for the points of one of these planes is $(ab)(ac)(cb)(ad)(bd)(cd)$ and the notation for the plane itself may be taken as $(abcd)$.

If the heptads are denoted by the 8 letters a,b,c,d,e,f,g,h, it will be found that the 2 planes $(abcd)(efgh)$ have but 1 point in common with the surface S and that these common points are the same. Hence the points of the surface S can be denoted by a double 4 letter notation, which will also be the notation for the lines of the 3-space. On page 63 the line coordinates of all the lines are obtained from those of the 3 lines (1,2,5), (1,3,9), (1,6,11), by use of the transformation T and its powers. The 8 letter notation for all the lines of the 3-space can be obtained from the notations for the 3 lines above, which are respectively (1278 + 3456), (1357 + 2468), (1467 + 2358), by use of the transformation on the heptads, which is isomorphic with T, i.e., (17654)(283).

The Determination of an Eight Letter Notation for the Points and Planes of the 3-space. A point of the 3-space is determined by the 7 lines which pass through it, and these correspond to the 7 points of a plane in the 5-space.

A plane of the 3-space is determined also by the 7 lines which lie in it, and these correspond in the 5-space to the 7 points of a plane. To each of the points of the 5-space there belongs a double four letter notation. Hence a point of the 3-space will be denoted by 14 sets of 4 numbers. The eight letter notation for the point $1 = (1,1,1,1)$ of the 3-space is 1278 + 1458 + 1234 + 1357 + 1256 + 1368 + 1467, 3456 + 2367 + 5678 + 2468 + 3478 + 2457 + 2358 and the notation for the point i is obtained from this by the ith power of T, where $T = (17654)(283)$.

These 14 sets of 4 numbers constitute the planes of a finite Euclidian geometry, where the numbers are considered as points; there are 4 points to a plane and 2 points to a line. This same Euclidian geometry can be obtained from the $PG(3,2)$ by striking out all the points of a plane, which will cut out 3 points from each plane. Another statement of the above is that the 14 sets of 4 numbers constitute a quadruple system,* for when any 3 numbers are given there is always a fourth determined.

The proof is as follows. When 3 numbers as a, b, c, are given this determines a line in the 5-space; this line determines a congruence in the 3-space whose 5 lines have the notation

$$abcd + efgh,\ abce + dfgh,\ abcf + degh,\ abcg + defh, \text{ and } abch + defg.$$

But through any point of the 3-space there is but one line belonging to a congruence, hence the fourth letter belonging to abc in the notation for this line is determined when this line is given.

The group, which leaves a plane of the 3-space or a point of the 3-space fixed, is of order $14 \cdot 12 \cdot 8 = 1344$ and this is the order of the group of the Euclidian geometry and of the quadruple systems on 8 letters.

There are 15 quadruple systems related to the planes of the 3-space and 15 related to the points of the 3-space. The points of the 3-space are transformed into the planes and vice versa by the odd transformations of the symmetric group on 8 letters, hence the 2 sets of 15 quadruple systems are conjugate under the symmetric group, while the members of each set of 15 are conjugate under the alternating group, which changes points into points and planes into planes.

The 8 letter notation for the points of the 3-space enables one to calculate the corresponding transformations of the $L.\,H.\,G.\ \overset{15}{\underset{2}{|\underline{8}}}$ and the $G\overset{8}{\underset{2}{|\underline{8}}}$ immediately. The method of passing from the linear homogeneous group to the alternating group has already been given on page 11. To pass in the reverse direction it is only necessary to determine into what points the vertices of the fundamental tetrahedron are transformed. This gives a method, which does not require the direct use of a table of corresponding transformations.†

Applications to Kirkman's School Girl Problem.‡ The problem is "to arrange 15 school girls in parties of 3 for 7 consecutive day's walks, so that no 2 girls may walk together more than once during the 7 days."

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* E. H. Moore, loc. cit.

† Such a table is given by L. E. Dickson. *Mathematische Annalen*, vol. 54 (1901), page 564.

‡ See Ball, *Mathematical Recreations and Problems*, page 89, for numerous references to the problem.

A congruence of the 3-space as

5, 10, 15		1467 + 2385
1, 6, 11		1567 + 2384
2, 7, 12	or	4567 + 2381
3, 8, 13		1456 + 2387
4, 9, 14		1457 + 2386

(the latter in the 8 letter notation) is obtained by polarizing for the surface S with respect to the 3 points of a line in the 5-space which is wholly off the surface S, and taking the points of the 5-space which are common to the surface S and the 3 polar 4-spaces; these common points represent the lines of a congruence in the 3-space. The above congruence is obtained by polarizing with respect to the 3 points

(1,0,1,0,0,1)		28
(1,1,0,0,1,0)	or	38
(0,1,1,0,1,1)		23

(the latter in the 8 letter notation). The group of a congruence is thus composed of all members of the alternating group which permute 3 letters among themselves, hence is of the order $5 \cdot 3/2 = 360$. A day of the school girl problem is evidently a congruence.

A week's solution consists of 7 congruences which do not have a line in common. The 5 lines of a congruence can be written in the 8 letter notation $abcd + efgh$, $abce + dfgh$, $abcf + degh$, $abcg + defh$, and $a\,ch + defg$, i. e., they are given by $abcx + defgh/x$, $x = d, e, f, g, h$, and the congruence is determined by polarizing with respect to the 3 points ab, bc, ca, of a line in the 5-space. There are 2 types of congruences having a line in common with the above congruence, I, $abdx + cefgh/x$, $x = c, e, f, g, h$, obtained by polarizing with respect to the points ab, bd, da, and II, $abcdh/x + efgx$, $x = a, b, c, d, h$, obtained by polarizing with respect to the points ef, fg, ge. Hence in order that 2 congruences shall have a line in common, the lines in the 5-space with respect to which we polarize must either have a point in common as ab or they must not belong to the same heptad. Hence the school girl problem consists of finding 7 lines in the 5-space which do not intersect and such that any 2 lines always have a heptad in common.

The 8 heptads give a complete solution of the problem. We take 7 of the heptads as (1234567) and form a $PG(2,2)$ with heptads as points, and the following triads as lines:

123, 145, 167, 347, 246, 257, 356.

Each of the sets of 3 numbers determines a line in the 5-space which is wholly off the surface S and no 2 of the 7 lines have a point in common and each two have a heptad in common. These 7 lines in the 5-space thus determine a solution of the problem.

There are 30 $PG(2,2)$ related to 7 letters, and since there are 8 heptads there are $30 \cdot 8 = 240$ solutions of the school girl problem belonging to this geometry. The 30 $PG(2,2)$ belonging to a set of 7 heptads are conjugate under the symmetric group and there are 2 sets of 15 such that the members of each set are conjugate among themselves under the alternating group. Hence there are 2 sets of 120 solutions each, the solutions of each set are permuted among themselves by the projective collineation group * and one set is transformed into the other by a polarity.

If we apply the cyclic transformation (12345678) to the above $PG(2,2)$ we obtain 8 solutions, which do not have a congruence or day in common.† These 8 solutions embrace all 56 congruences. By a transformation of period 15 we can obtain 120 solutions from these 8 and by means of a duality all the 240 are obtained.

Each solution is invariant under a group of order 168, since it is a $PG(2,2)$ on 7 heptads. From this $PG(3,2)$ we derive 240 solutions. There are $2|\underline{15}/8$ equivalent spaces which are conjugate with this space, hence there are $240 \cdot 2 \cdot |\underline{15}/8$ solutions of the school girl problem to be obtained from these spaces. J. Power has shown that this is the number of possible solutions, hence each school girl solution is related to a $PG(3,2)$.‡

Application to the Equations of the Eighth and Lower Degrees. A particular expression,§

$$v = x_1x_2x_7x_8 + x_1x_4x_5x_8 + x_1x_2x_3x_4 + x_1x_3x_5x_7 + x_1x_2x_5x_6 + x_1x_3x_6x_8 + x_1x_4x_6x_7$$
$$+ x_3x_4x_5x_6 + x_2x_3x_6x_7 + x_5x_6x_7x_8 + x_2x_4x_6x_8 + x_3x_4x_7x_8 + x_2x_4x_5x_7 + x_2x_3x_5x_8,$$

* E. H. Moore, loc. cit., page 441.

† E. H. Moore, loc. cit., page 443.

‡ J. Power, On the Problem of the Fifteen School Girls. *Quarterly Journal of Mathematics*, vol. 8 (1867), pp. 236-251.

§ Mathieu, *Journal de mathématiques pures et appliqués*, vol. 6 (1861), pp. 241—323. See page 291.

which is the notation for a point or a plane in the 3-space and is invariant under a group of substitutions of order $14 \cdot 12 \cdot 8 = 1344$, has been used to reduce the general equation of the eighth degree to a special one whose Galois group is of order 1344.* The expression v has 15 conjugates under the alternating group and hence is the root of an equation of the fifteenth degree whose coefficients can be rationally expressed in terms of the coefficients of the original equation and of the square root of the discriminant. On adjoining a root v of this equation of the fifteenth degree and adjoining the square root of the discriminant, the general equation of eight degree reduces to a particular one with the Galois group of order 1344.

The equation of the seventh degree may be considered by allowing one of the heptads to be fixed. A function which plays a similar role to the one above for the equation of the seventh degree is

$$v = x_1x_5x_7 + x_2x_6x_1 + x_3x_7x_2 + x_4x_1x_3 + x_5x_2x_4 + x_6x_3x_5 + x_7x_4x_6.$$

This is the notation for a school girl solution and as has been shown is invariant under a group of order 168. Thus v has 15 conjugates under the alternating group. Hence on adjoining a root of an equation of the fifteenth degree and adjoining also the square root of the discriminant, the general equation of the seventh degree reduces to a special equation whose Galois group is of order 168.†

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* H. Weber, *Lehrbuch der Algebra*, vol. 2, page 377.

† H. Weber, loc. cit., page 540.